Two-Impulse Transfer vs One-Impulse Transfer: Analytic Theory

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This paper presents a mathematical proof that, if two coplanar elliptical orbits with a common focus intersect for some orientations, the optimum one-impulse transfer between the two orbits is still inferior to the two-impulse Hohmann transfer. Numerical results indicating this result are mentioned by L. Ting ("Optimum orbital transfer by impulses," ARS J., vol. 30, pp. 1013–1018, 1960), and the proof for the special case of an intersecting elliptical and circular orbit is given by J. M. Horner ("Optimum impulsive orbital transfers between coplanar orbits," ARS J., vol. 32, pp. 1082–1089, 1962).

In the concluding paragraph of Ting (1),[2] the following statement appears:

> If the two (coplanar) orbits (with common focus) can intersect each other, orbital transfer can be accomplished by either one or two impulses. By comparison of numerical results, the sum of two impulses in the optimum transfer is always less than the corresponding optimum single impulse. An analytic proof of this statement is desirable.

In the present paper, an analytic proof that the one-impulse transfer is always inferior to the two-impulse Hohmann transfer will be given. By the two-impulse Hohmann transfer, it is meant that the two orbits and the transfer trajectory are coplanar with the same major axis and oriented in the same sense, and that the apogee of the transfer trajectory coincides with that of the orbit with longer apogee distance, whereas the perigee of the transfer trajectory coincides with that of the other orbit. The results of the present paper have been proved by Horner (5)[3] for the special case of intersecting circular and elliptic orbits. A method of dividing the total impulse into a component perpendicular to the radius vector and another component perpendicular to the axis of the conic will be used. In a forthcoming paper (4), the methods and results of this paper will be used to give a proof of the commonly assumed but nonetheless difficult to prove result that, of all two-impulse transfers, the Hohmann transfer is optimum (see Ref. 3, concluding paragraph).

Whittaker Theorem

The following theorem is stated and proved in Ref. 2, p. 89. (Actually, the theorem also follows from Ref. 1, Eqs. [2a] and [2b].)

It can be shown that the velocity at any point on an elliptic orbit can be resolved into a component $y = \mu/h$ perpendicular to the radius vector and a component $x = \mu e/h$ perpendicular to the axis of the conic, each of these components being constant.

The equation of a conic

$$(1/r) = (1/p)(1 + e \cos f) \quad [1]$$

is partially related to dynamics through the equation

$$p = h^2/\mu = a(1 - e^2) \quad [2]$$

where

r = distance from common focus
f = true anomaly
μ = mass of earth times universal gravitational constant
h = angular momentum
p = semilatus rectum
e = eccentricity
a = semimajor axis

It follows from [1] and [2] that

$$\mu/r = (\mu^2/h^2)(1 + e \cos f) = y^2 + yx \cos f \quad [3]$$

Throughout this paper, expressions such as $yx \cos f$ will always mean $y \cdot x \cdot (\cos f)$. It is also obvious that, besides a and e determining x and y, conversely x and y determine a and e. For the present purposes, it will be most convenient to describe an ellipse using x and y.

Ellipses Having Parallel and Equally Oriented Major Axes

It will also be convenient to introduce the notation

$$k = \mu/r_{\text{perigee}} = y^2 + xy \quad [4]$$

$$l = \mu/r_{\text{apogee}} = y^2 - xy \quad [5]$$

Eq. [4] in the x, y plane is a family of hyperbolas having as asymptotes the lines $y = 0$ and $x = -y$, whereas Eq. [5] is a family of hyperbolas having asymptotes $y = 0$ and $x = y$. Note that the region of interest in this instance is for $y > x > 0$ (see Fig. 1).

By adding and subtracting Eqs. [4] and [5], x and y can be found in terms of k and l; thus

$$y = [(k + l)/2]^{1/2} \quad [6]$$

$$x = (k - l)/2y = (k - l)/[2(k + l)]^{1/2} \quad [7]$$

Thus, knowing the apogee and perigee of an ellipse, Eqs. [4–7] enable one to compute the velocity at any point on the orbit.

In order to understand the physical meaning of Fig. 1 better, imagine that two ellipses are given, designated by the points x_1,y_1 and x_2,y_2, and that it is desired to go from one to the other by means of a third ellipse that has the same perigee as the first and the same apogee as the second. Clearly, this is the ellipse designated by x_3,y_3 in Fig. 1.

Assume for the moment that all the ellipses have parallel and equally oriented major axes. If the velocity vector is resolved into components perpendicular and parallel to the radius vector, the velocity at any point on the orbit is found to be $[(y + x \cos f)^2 + (x \sin f)^2]^{1/2}$; hence the velocity at perigee is $y + x$, and at apogee it is $y - x$. Thus the total change of velocity V_{12} in first going from orbit 1 to 3 and then from 3 to 2 will be

$$V_{12} = |y_1 + x_1 - y_3 - x_3| + |y_3 - x_3 - y_2 + x_2| \quad [8]$$

Conversely, if one wishes to transfer via an orbit having the same apogee as x_1,y_1 and perigee as x_2,y_2, one would use

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[1] Member of the Technical Staff.
[2] Numbers in parentheses indicate References at end of paper.
[3] The author wishes to thank the reviewer for calling this reference to his attention.

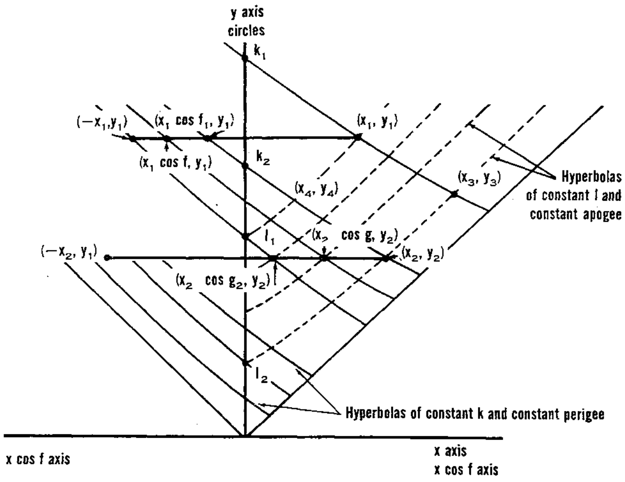

Fig. 1 The x, y plane

the ellipse x_4,y_4 with the total change of velocity

$$V_{12}' = |y_1 - x_1 - y_4 + x_4| + |y_4 + x_4 - y_2 - x_2|$$

Since in this particular case $l_2 < l_1$, one sees that the apogee of orbit 2 is greater than that of orbit 1, and one can prove in accordance with Ting (1) that $V_{12}' > V_{12}$. (This result is not required in the present paper.)

From the relations $l_2 < l_1 < k_2 < k_1$, it is clear that this configuration represents the case labeled by Ting (1) "Initial and final orbits intersect each other at certain orientations."

Ellipses Coplanar but Arbitrarily Oriented

In order to take care of the case of two orbits intersecting at various orientations, return to Eq. [3]. From this equation it follows that, if orbit 1 is to intersect orbit 2, when orbit 1 has true anomaly f and orbit 2 has true anomaly g, one must have

$$y_1^2 + y_1(x_1 \cos f) = y_2^2 + y_2(x_2 \cos g) = k \qquad [9]$$

Hence, if one plots curves similar to the constant perigee curve but now labels the x axis as $x \cos f$, this case can also be treated (see Fig. 1). The k defined in Eq. [9] is a generalization of the k defined in Eq. [4], but, since the two quantities represent the same thing in this discussion, the same letter has been used for both.

Thus, in Fig. 1, if $180° > f > f_1$, there exists a g such that $g_2 > g > 0$ and such that a one-impulse transfer can take place between the two orbits at a distance r_0 where $y_1^2 + y_1(x_1 \cos f) = y_2^2 + y_2(x_2 \cos g) = \mu/r_0$.

Resolving the velocity vector in components parallel and perpendicular to the radius, one finds that the velocity change is

$$\Delta V = [(y_1 + x_1 \cos f - y_2 - x_2 \cos g)^2 + (x_1 \sin f - x_2 \sin g)^2]^{1/2} \qquad [10]$$

The values of the true anomaly are restricted to lie between 0° and 180°. This has no effect on the first term of Eq. [10] but assures that the members in the second term always subtract. This minimizes [10] for any values of the true anomaly satisfying [9].

The problem of minimizing the one-impulse transfer between intersecting orbits is that of minimizing Eq. [10] subject to the constraint [9]. In this formulation, when the minimum does not occur at an end point, it turns out to be the root of a fifth-degree equation. This result will not be discussed in this paper.

The problem of showing that the one-impulse transfer is less efficient than the two-impulse Hohmann transfer in the case of intersecting orbits amounts in the present formulation to proving that

$$V_{12} \leqslant \Delta V \qquad [11]$$

where V_{12} is defined in Eq. [8] and ΔV is defined in Eqs. [9] and [10]. This will be done in the next section. Actually, the equality sign in Eq. [11] will occur only in the degenerate cases when orbits 1 and 2 have the same apogee distance or perigee distance, because in this case the optimum one-impulse transfer and the two-impulse Hohmann transfer become the same thing. This will be obvious in the proof.

Proof of Eq. [11]

Case 1: Orbits Always Intersect

Orbit 2 is always assumed to have the largest apogee. One first treats the case when the two orbits always intersect, $l_2 < l_1 < k_1 < k_2$ (see Fig. 2). In order to remove the absolute value signs in Eq. [8], one must see which is the biggest term.

For this purpose, Eqs. [4] and [5] are rewritten as

$$y + x = k/y \qquad [4a]$$

$$y - x = l/y \qquad [5a]$$

From Fig. 2, $y_3 < y_1$, $y_3 < y_2$. (This also follows from Eq. [6].) Combining this with [4a] and [5a], one has for the case of always intersecting orbits

$$V_{12} = 2y_3 - (y_1 + y_2) + (x_2 - x_1) \qquad [8a]$$

Thus the proof of Eq. [11] reduces to proving

$$V_{12}^2 = [(x_2 - x_1) - (y_1 + y_2 - 2y_3)]^2 \leqslant [(y_1 + x_1 \cos f) - (y_2 + x_2 \cos g)]^2 + [x_1 \sin f - x_2 \sin g]^2 = (\Delta V)^2 \qquad [11a]$$

Proof: From [8a], it follows that $x_2 \geqslant x_1$. From [9], it follows that $y_1 + x_1 \cos f \geqslant y_2 + x_2 \cos g$ if $y_2 \geqslant y_1$. Hence also $x_1 \cos f - x_2 \cos g \to 0$. Below, this subcase is considered. The same reasoning applies to the subcase $y_1' \to y_2$ (see Fig. 2, case with primes).

In the present case, one always has

$$(y_2 - y_1) \leqslant (y_1 + y_2 - 2y_3) \qquad [12]$$

since the left side is obtained by replacing y_3 by y_1. Furthermore, one obviously has

$$(x_2 - x_1)^2 = x_2^2 + x_1^2 - 2x_1x_2 \leqslant x_1^2 + x_2^2 - 2x_1x_2 \cos(f - g) \qquad [13]$$

The result now easily follows from [12] and [13], since if $(x_2 - x_1) \leqslant (x_1 \cos f - x_2 \cos g)$, then

$$V_{12} = (x_2 - x_1) - (y_1 + y_2 - 2y_3) \leqslant (x_1 \cos f - x_2 \cos g) - (y_2 - y_1) \qquad [14]$$

Hence the first term on the right of [11a] is already larger than V_{12}^2. Conversely, if $(x_2 - x_1) \geqslant (x_1 \cos f - x_2 \cos g)$, one has

$$V_{12}^2 \leqslant [(x_2 - x_1) - (y_2 - y_1)]^2 = (y_2 - y_1)^2 - 2(x_2 - x_1)(y_2 - y_1) + (x_2 - x_1)^2 \leqslant (y_2 - y_1)^2 - 2(x_1 \cos f - x_2 \cos g)(y_2 - y_1) + x_1^2 + x_2^2 - 2x_1x_2 \cos(f - g) = (\Delta V)^2 \quad \text{QED}$$

Case 2: Orbits Intersect Each Other at Certain Orientations

Here $l_2 < l_1 < k_2 < k_1$, and now

$$V_{12} = [2x_3 - (x_1 + x_2)] - (y_1 - y_2) \qquad [8b]$$

with $y_1 > y_3 > y_2$ by Eq. [6].

For the present case one needs some estimates.

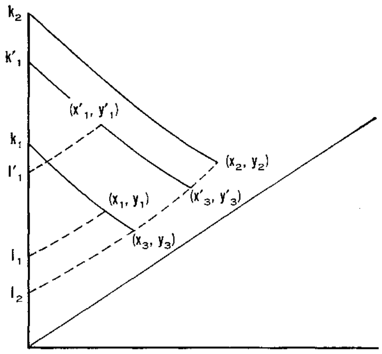

Fig. 2 The configuration in case 1

Estimate 1

If $H^* = \min H$, where $H = -2x_1x_2 \cos(f - g)$, then

$$H^* = -[(y_2x_2)^2 + (y_1x_1)^2 - (y_1^2 - y_2^2)^2]/(y_1y_2) \geqslant -2x_1x_2 \cos(f - g)$$

Proof: One first points out that in the present case $f \neq g$. To see this, one need only consider the plot of $x_1 \cos f$ vs $x_2 \cos g$. This is clearly the straight line of Fig. 3:

$$x_1^2 + y_1(x_1 \cos f) = y_2^2 + y_2(x_2 \cos g) \qquad [15]$$

From the inequalities

$$y_2^2 - y_2x_2 < y_1^2 - y_1x_1 < y_2^2 + y_2x_2 < y_1^2 + y_1x_1 \qquad [16]$$

it follows that there exist angles g_2, f_1 such that

$$y_2^2 + y_2x_2 \cos g_2 = y_1^2 - y_1x_1 \qquad [17]$$

$$y_1^2 + y_1x_1 \cos f_1 = y_2^2 + y_2x_2 \qquad [18]$$

for points A and B, respectively, of Fig. 3. Clearly, since the line [15] passing through the points A and B of Fig. 3 does not intersect the line $\cos f = \cos g$, the assertion follows.

Hence, the only vanishing of

$$dH/df = 2x_1x_2 \sin(f - g)(1 - dg/df) \qquad [19]$$

can occur at $dg/df = 1$.

On the other hand, it follows from [15] that

$$dg/df = (y_1x_1 \sin f)/(y_2x_2 \sin g) \qquad [20]$$

Substituting [20] into [19] and evaluating [19] at $f = f_1$ and $f = 180°$, one finds

$$(dH/df)_{f\to f_1} = \lim_{g\to 0} 2x_1x_2 \sin(f_1 - g)[1 - (y_1x_1 \sin f_1 y)/(y_2x_2 \sin g)] = -\infty$$

$$(dH/df)_{f=180°} = 2x_1x_2 \sin(180° - g_2) = 2x_1x_2 \sin g_2 > 0$$

Thus, in the interval $180° > f > f_1$, H assumes its minimum in the interior. It will be shown that there is only one point at which $dg/df = 1$; hence it is the true minimum.

It follows from Eqs. [20] and [15] that at the minimum (i.e., $dg/df = 1$)

$$y_1x_1 \sin f_{\min} = y_2x_2 \sin g_{\min} \qquad [21]$$

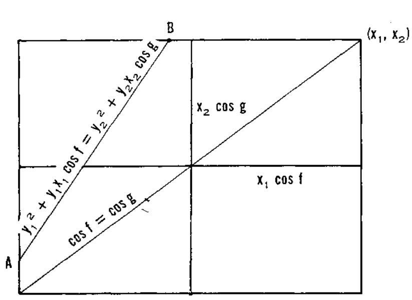

Fig. 3 Plot of $x_2 \cos g$ vs $x_1 \cos f$

$$(y_1^2 - y_2^2) + y_1x_1 \cos f_{\min} = y_2x_2 \cos g_{\min} \qquad [22]$$

Squaring and adding [21] and [22], one obtains

$$2(y_1^2 - y_2^2)y_1x_1 \cos f_{\min} = (y_2x_2)^2 - (y_1x_1)^2 - (y_1^2 - y_2^2)^2 \qquad [23]$$

Hence, there is only one possible $f_{\min}$.

Furthermore, from [21–23] it follows that

$$\begin{aligned} H^* &= -2x_1x_2[\cos g_{\min} \cos f_{\min} + \sin g_{\min} \sin f_{\min}] \\ &= -2x_1x_2\{(\cos f_{\min})[(y_1^2 - y_2^2) + y_1x_1 \cos f_{\min}]/(y_2x_2) + (\sin f_{\min})(y_1x_1 \sin f_{\min})/(y_2x_2)\} \\ &= -2[(y_1^2 - y_2^2)(y_1x_1 \cos f_{\min}) + (y_1x_1)^2]/(y_1y_2) \\ &= -[(y_2x_2)^2 + (y_1x_1)^2 - (y_1^2 - y_2^2)^2]/(y_1y_2) \qquad \text{QED} \end{aligned}$$

Estimate 2

If, as in the present case, $l_2 \leqslant l_1 \leqslant k_2 \leqslant k_1$, then $F(k_2) - F(k_1) \geqslant 0$, where

$$F(k) = (k - l_2)/[2(k + l_2)]^{1/2} - (k - l_1)/[2(k + l_1)]^{1/2}$$

Proof:

$$F(k) = (k/2)^{1/2}[(1 - l_2/k)/(1 + l_2/k)^{1/2} - (1 - l_1/k)/(1 + l_1/k)^{1/2}]$$

$$= \left(\frac{k}{2}\right)^{1/2} \int_{l_1/k}^{l_2/k} f'(x)dx$$

with

$$f(x) = (1 - x)/(1 + x)^{1/2}$$

Now the proof consists merely of showing $F'(k) \leqslant 0$; however, the algebra gets a little involved:

$$F'(k) = \left[\frac{1}{2(2k)^{1/2}}\right] \int_{l_1/k}^{l_2/k} f'(x)dx - \left(\frac{k}{2}\right)^{1/2} \times \left[\left(\frac{l_2}{k^2}\right) f'\left(\frac{l_2}{k}\right) - \left(\frac{l_1}{k^2}\right) f'\left(\frac{l_1}{k}\right)\right]$$

$$= \left(\frac{1}{2k}\right)^{1/2} \left[xf'(x) - \frac{f(x)}{2}\right]\Bigg|_{l_2/k}^{l_1/k}$$

$$= \left(\frac{1}{2k}\right)^{1/2} \int_{l_2/k}^{l_1/k} g'(x)dx \qquad [24]$$

where $g(x) = xf'(x) - f(x)/2$. One easily finds

$$g'(x) = 3(x - 1)/[4(1 + x)^{5/2}] \leqslant 0 \qquad [25]$$

Hence, the integrand is negative, and the estimate now follows.

Estimate 3

Let $l_2 \leqslant l_1 \leqslant k_2 \leqslant k_1$; let x_3,y_3 be the ellipse whose apogee is the same as x_2,y_2 and whose perigee is the same as x_1,y_1; and let $y_1^2 + y_1x_1 \cos f_1 = y_2^2 + y_2x_2$. Then the following estimate is valid:

$$(x_1 - x_3) \geqslant (y_3/y_2)[x_1(1 + \cos f_1)/2] - x_2$$

Proof: In terms of k's and l's, one has

$$y_1 + x_1 \cos f_1 = k_2/y_1$$

Hence

$$x_1 + x_1 \cos f_1 = (k_1 + k_2)/y_1 - 2y_1 = [k_1 + k_2 - (k_1 + l_1)]/y_1 = (k_2 - l_1)/y_1$$

Now one can define G and H by the formulas

$$(x_1 - x_3) = G(k_1, l_1, l_2) = \{(k_1 - l_1)/[2(k_1 + l_1)]^{1/2} - (k_1 - l_2)/[2(k_1 + l_2)]^{1/2}\}$$

$$(y_3/y_2)x_1(1 + \cos f_1)/2 - x_2 = H(k_1, k_2, l_1, l_2) = \{(k_1 + l_2)/(k_2 + l_2)^{1/2}\}(k_2 - l_1)/[2(k_1 + l_1)]^{1/2} - (k_2 - l_2)/[2(k_2 + l_2)]^{1/2}$$

From these formulas it follows that

$$G(k_2, l_1, l_2) = H(k_2, k_2, l_1, l_2) \quad [26]$$

Hence, the estimate will follow if it can be proved that

$$\partial G(k, l_1, l_2)/\partial k \geqslant \partial H(k, k_2, l_1, l_2)/\partial k \quad [27]$$

with $k > k_2$.

Before proceeding, note the obvious result that if

$$L(k) = (k - l_1)/(k + l_2)^{1/2} \quad [28]$$

then

$$L'(k) \geqslant 0 \qquad L(k) \geqslant L(k_2) \qquad k \geqslant k_2 \quad [29]$$

Now using [24], [25], [28], and [29], one finds

$$4\,\frac{\partial G(k, l_1, l_2)}{\partial k} = -F'(k) = \left[\frac{3}{(2k)^{1/2}}\right] \int_{l_2/k}^{l_1/k} \frac{(1 - x)}{(1 + x)^{5/2}}\,dx \geqslant \left[\frac{3}{(2k)^{1/2}}\right]\left[\frac{(1 - l_1/k)}{(1 + l_1/k)^{1/2}}\right] \int_{l_2/k}^{l_1/k} \frac{dx}{(1 + x)^2}$$

$$= \frac{[3/(2k)^{1/2}](1 - l_1/k)(l_1 - l_2)}{(1 + l_1/k)^{1/2}k(1 + l_1/k)(1 + l_2/k)}$$

$$= \frac{(3/2^{1/2})L(k)(l_1 - l_2)}{[(k + l_1)^3(k + l_2)]^{1/2}}$$

$$\leqslant \frac{2^{1/2}L(k_2)(l_1 - l_2)}{[(k + l_1)^2(k + l_2)]^{1/2}} = 4\,\frac{\partial H(k, k_2, l_1, l_2)}{\partial k} \qquad \text{QED}$$

One is now ready to prove formula [11]. In the present case, it assumes the form

$$V_{12}^2 = [2x_3 - (x_1 + x_2) - (y_1 - y_2)]^2 \leqslant [(y_1 + x_1 \cos f) - (y_1 + x_2 \cos g)]^2 + [x_1 \sin f - x_2 \sin g]^2 = (\Delta V)^2 \quad [11b]$$

Just as the proof of [11a] followed from [12] and [13], one easily sees that [11b] will follow from the following two formulas (see Estimate 1):

$$(y_1 - y_2) = (y_1 - y_2) \quad [12a]$$

$$[2x - (x_1 + x_2)]^2 \leqslant x_1^2 + x_2^2 - (1/y_1y_2)[(y_2x_2)^2 + (y_1x_1)^2 - (y_1^2 - y_2^2)^2]$$

$$\leqslant x_1^2 + x_2^2 - 2x_1x_2 \cos(f - g) \quad [13a]$$

Eq. [12a] is of course trivial, and actually [13a] is just Estimate 3. To see this, proceed as follows.

Write the difference of the first two expressions in [13a] as

$$Q = x_1^2 + x_2^2 - (1/y_1y_2)[(y_2x_2)^2 + (y_1x_1)^2 - (y_1^2 - y_2^2)] - [2x_3 - (x_1 + x_2)]^2$$

Expanding the last term, one obtains

$$Q = 4x_3(x_1 + x_2 - x_3) - 2x_1x_2 - (1/y_1y_2)[(y_2x_2)^2 + (y_1x_1)^2 - (y_1^2 - y_2^2)^2] \quad [30]$$

Now, subtracting

$$y_1^2 + y_1x_1 = y_3^2 + y_3x_3$$

from

$$y_2^2 - y_2x_2 = y_3^2 - y_3x_3$$

one obtains

$$y_1^2 - y_2^2 = 2y_3x_3 - [(y_1x_1) + (y_2x_2)] \quad [31]$$

Substituting [31] in [30], one finds

$$Q = 4x_3(x_1 + x_2 - x_3) - (4y_3x_3/y_1y_2)(y_1x_1 + y_2x_2 - y_3x_3)$$

Now, substituting

$$(y_1^2 - y_2^2) = y_2x_2 - y_1x_1 \cos f_1$$

in [31], one obtains

$$y_3x_3 = (y_2x_2) + (y_1x_1)(1 - \cos f_1)/2$$

Substituting this for y_3x_3 in Q, one finally obtains

$$Q/4x_3 = (x_1 + x_2 - x_3) - (y_3/y_2)x_1(1 + \cos f_1)/2 \geq 0$$

by Estimate 3. This implies [13a] and thus completes the proof.

Conclusion

A mathematical proof has been presented herein to show that the Hohmann transfer is superior to the optimum one-impulse transfer for two coplanar elliptic orbits with a common focus which can intersect for some orientations. The methods used in this paper can be extended to prove that the Hohmann transfer is optimum for all two-impulse transfers; at the present time this is only proved for "nearly Hohmann-type" transfers (see Refs. 3 and 4).

References

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